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Zeros of the Hurwitz zeta function in the interval $(0, 1)^*$

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Abstract

We first give a condition on the parameters s, w under which the Hurwitz zeta function $\zeta(s, w)$ has no zeros and is actually negative. As a corollary we derive that it is nonzero for $w \geq 1$ and $s \in (0, 1)$ and, as a particular instance, the known result that the classical zeta function has no zeros in $(0, 1)$.

1 Statement and proof of the results

The Hurwitz zeta function is classically defined for $\Re(s) > 1$ as

$$\zeta(s, w) \doteq \sum_{n=0}^{\infty} (n + w)^{-s},$$

with w being a positive real number, and it can be continued analytically to the whole s-plane, except for a pole in 1 (see e.g. [2, Proposition 9.6.6]). Sometimes the definition is extended by letting w be a complex number, while in other situations w is only restricted to be a real number in $(0, 1]$. Notice in fact the following relation:

$$\zeta(s, w) = \zeta(s, w + 1) + w^{-s}$$

which follows by considering $n + 1$ instead of n.

We set as usual $\sigma = \Re(s)$. The following theorem is meant to add a new result to what is already known about its zeros (see e.g. [4]).

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Theorem 1.1. *Suppose that* $1 - w \leq \sigma$*. Then* $\zeta(s, w)$ *is negative and in particular nonzero for* $s \in (0, 1)$*.*

Proof. As a first step we derive a representation for the Hurwitz zeta function through the Euler-Maclaurin summation formula, in analogy with the one derived for the classical Riemann zeta function (see e.g [3, chapter 6]).

Namely, the following Euler-Maclaurin summation formula

$$\sum_{n=N}^{M} f(n) = \int_{N}^{M} f(x)dx + \frac{1}{2}[f(M) + f(N)] + \frac{B_2}{2} f'(x)\Big|_N^M + \frac{B_4}{4!} f^{(3)}(x)\Big|_N^M \ldots,$$

where $B_2, B_4, \ldots$ are Bernoulli numbers, is applied to $f(n) = (n + w)^{-s}$ letting M tend to ∞, to obtain, for $\Re(s) > 1$,

$$\zeta(s, w) = \sum_{n=0}^{N-1} (n + w)^{-s} + \frac{(N + w)^{1-s}}{s - 1} + \frac{1}{2}(N + w)^{-s} + O((N + w)^{-\sigma-1}).$$

One then argues that this formula is in fact true for all $\mathbb{C} \backslash \{1\}$, as explained for the zeta function ([3, section 6.4]), by considering an expression for the rest in integral form and its correspondent halfplane of convergence (see also [2, Proposition 9.6.7]).

The formula above implies that, if we restrict to $\Re(s) > 0$,

$$\zeta(s, w) = \lim_{N \to \infty} \left(\sum_{n=0}^{N} (n + w)^{-s} - \frac{(N + w)^{1-s}}{1 - s} \right).$$

Now, by the triangular inequality,

$$\left| \sum_{n=0}^{N} (n + w)^{-s} \right| \leq \sum_{n=0}^{N} (n + w)^{-\sigma} < w^{-\sigma} + \int_{0}^{N} (x + w)^{-\sigma} dx = w^{-\sigma} + \frac{(N + w)^{1-\sigma}}{1 - \sigma} - \frac{w^{1-\sigma}}{1 - \sigma}$$

(in the case of real s, which is what will interest us now, the first comparison is in fact an equality)

which we write as

$$\frac{(N + w)^{1-\sigma}}{1 - \sigma} + \frac{(1 - \sigma)w^{-\sigma} - w^{1-\sigma}}{1 - \sigma}.$$

Suppose first that $(1 - \sigma)w^{-\sigma} < w^{1-\sigma}$, which is the same as $1 - w < \sigma$. Since $1 - \sigma$ is positive for $\sigma < 1$, then the expression above is less than

$$\frac{(N + w)^{1-\sigma}}{1 - \sigma} - a,$$

where a is a fixed positive quantity.

Therefore, in $(0, 1)$ and for every N, $\left|\sum_{n=0}^{N} (n + w)^{-s}\right|$ which is also $\sum_{n=0}^{N} (n + w)^{-s}$ or $\sum_{n=0}^{N} (n + w)^{-\sigma}$ is less than $\left|\frac{(N+w)^{1-s}}{1-s}\right| = \frac{(N+w)^{1-s}}{1-s} = \frac{(N+w)^{1-\sigma}}{1-\sigma}$ by more than a fixed positive quantity. Then in the limit the difference is nonzero, and in fact negative.

For the case $1-w=\sigma$, notice that we would not be able to conclude if, as N goes to ∞, the difference between $\sum_{n=0}^{N}(n+w)^{-\sigma}$ and $w^{-\sigma}+\int_0^N (x+w)^{-\sigma}dx$ could decrease and tend to 0. But this is not the case, since it is actually increasing, as the approximation with the integral is accumulating error as N gets bigger. The thesis then follows also in this scenario. □

Corollary 1.2. *Suppose $w \geq 1$. Then $\zeta(s,w)$ has no zeros for $s \in (0,1)$.*

A particular case is the following well known result (see e.g. [1, chapter 13]):

Corollary 1.3. *The Riemann zeta function $\zeta(s)$ has no zeros in $(0,1)$.*

Proof. In fact $\zeta(s)=\zeta(s,1)$. □

Remark 1.4. It is well known that the Dirichlet L-function, defined for $\Re(s) > 1$ as $L(s,\chi) \doteq \sum_{n=1}^{\infty}\frac{\chi(s)}{n^s}$, where χ is a character (mod q), can be represented through a sum of Hurwitz zeta functions (see e.g [2] or [1]) in the following way:

$$L(s,\chi)=q^{-s}\sum_{a=1}^{q}\chi(a)\zeta(s,a/q).$$

The Extended Riemann Hypothesis conjectures that the Dirichlet L-function $L(s,\chi)$, for a primitive character χ, has no zeros with real part different from $1/2$ in the critical strip; and its strong version says that $L(1/2,\chi)$ is always nonzero too: see also [2, section 10.5.7]).

We leave then open for investigation to see whether our main result could help to find new zero-free regions for this class of functions, for example to show that the Dirichlet L-functions are also nonzero on the real axis between 0 and 1, which would thus establish a weaker version of the Extended Riemann Hypothesis, though stronger in including the point $1/2$.

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